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Implicit Finite Difference Solution of 1D Nonlinear Porous Medium Equation via Four-Point EGSOR with Newton Iteration

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ABSTRACT

Objective – Porous medium equation (PME) has a great practical in fluid flow, heat transfer and population dynamics. The nonlinearity in this equation makes it interesting in the development of nonlinear analytical and numerical tools in pure and applied mathematics and sciences. This paper proposes a Four-Point EGSOR with Newton iteration to solve the 1D PME problems.
Methodology/Technique – The reliability of proposed method is illustrated. The formulation and implementation of the proposed method are also presented.
Findings – The numerical results showed that the Four-Point EGSOR with Newton iteration requires less number of iterations and computational time in obtaining the numerical solution to the 1D PME problems.
Novelty – With these results, it can be said that the Four-Point Newton-EGSOR iterative method can be a promising numerical method in tackling nonlinear differential equation problems. To enhance the rate of convergence of the current method, in future work, this study will investigate the application of MSOR as in Sulaiman et al. (2012).

Type of Paper: Empirical

Keywords: Porous medium equation; implicit finite difference; Newton; Explicit Group; SOR.

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1. Introduction

The general form of the one-dimensional porous medium equation (1D PME) can be written as (Vazquez, 2006)

$$\frac{du}{dt} = K\frac{d}{dx}\left(u^m \frac{du}{dx}\right), \quad u(x,t), \quad x \in (a,b), \quad 0 < t < T \tag{1}$$

where K and m are real numbers.

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Eq. (1) is known in Soviet literature as the equation of Newtonian polytrophic filtration which can be reduced into the classical heat equation when the coefficient m is equal to 1. Eq. (1) get its name because of its description of the flow of an ideal gas in a porous medium where the unknown u is the density of the gas. In addition, the important constant of m represents the behaviour of the process (Gilding and Kersner, 2004). Besides that, Eq. (1) appears in the heat transfer problem with temperature-dependent thermal conductivity, the filtration of an incompressible fluid through a porous medium and the spread of biological populations. These applications have showed the emerging importance of the PME in investigating the nonlinear phenomena (Vazquez, 2006).

In fact, there has been a few methods proposed in order solve the Eq. (1) such as Variational Iterative Method (Sadighi and Ganji, 2007; Wazwaz, 2007), Adomian Decomposition Method (Pamuk, 2005; Wazwaz, 2005) and Homotopy Perturbation Method (Chun, 2009). However, none of these methods consider the efficiency of computing the approximate solutions. Inspired by these literatures, this paper proposes a Four-Point EGSOR with Newton iteration or also can be known as the Four-Point Newton-EGSOR iterative method to solve the Eq. (1). Actually, the proposed method is a combination between a Four-Point EGSOR iteration (Saudi and Sulaiman, 2012) and the Newton method that is used to handle the nonlinearity of the problem.

In this paper, implicit finite difference scheme is used in order to develop the finite difference approximation equation to the 1D PME. Nonlinear system that is formed at each time level can be linearized by using the Newton method and then transforms it into the corresponding linear system. The desired numerical solutions can be obtained by solving the linear system via the Four-Point EGSOR method. In this paper also, four examples of the 1D PME problems are chosen in order to verify the efficiency of the proposed method. The numerical results obtained by using the Four-Point Newton-EGSOR method will be compared with the Four-Point Newton-EG and Newton-Gauss-Seidel (GS) iterative methods.

Before applying the implicit finite difference scheme, let us consider the solution domain x and the time T be partitioned uniformly with equal sized of Δx and Δt respectively as

$$\Delta x = \frac{(b-a)}{d}, \quad \Delta t = \frac{T}{s}. \tag{2}$$

2. Methodology

2.1 Implicit Finite Difference Approximation Equation

Now, to formulate the implicit finite difference approximation equation to Eq. (1), Fig. 1 will be used to facilitate the development of computational molecules to the solution domain.

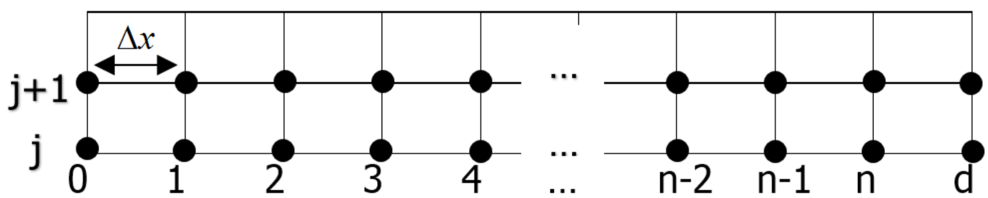

Figure 1. Finite grid network

To begin, Eq. (1) at first be rewritten as

$$\frac{du}{dt} = K\left[u^m \frac{d^2u}{dx^2} + mu^{m-1}\left(\frac{du}{dx}\right)^2\right]. \tag{3}$$

and then, Eq. (3) is discretized into an implicit finite difference approximation equation as

$$u_{i,j+1} - \alpha u_{i,j+1}^{m} u_{i+1,j+1} + 2\alpha u_{i,j+1}^{m+1} - \alpha u_{i,j+1}^{m} u_{i-1,j+1} \\ - \beta m u_{i,j+1}^{m-1} u_{i+1,j+1}^{2} + 2\beta m u_{i,j+1}^{m-1} u_{i+1,j+1} u_{i-1,j+1} - \beta m u_{i,j+1}^{m-1} u_{i-1,j+1}^{2} = u_{i,j} \tag{4}$$

for $i = 1, 2, 3..., n$ and $j = 0, 1, 2, ..., s$, where

$$\alpha = \frac{K\Delta t}{\Delta x^2}, \quad \beta = \frac{K\Delta t}{4\Delta x^2}.$$

Next, nonlinear function f is defined for each interior grid point in Fig. 1 that is given by

$$f_i\left(\underset{\sim}{u}_{j+1}\right) = u_{i,j+1} - \alpha u_{i,j+1}^{m} u_{i+1,j+1} + 2\alpha u_{i,j+1}^{m+1} - \alpha u_{i,j+1}^{m} u_{i-1,j+1} \\ - \beta m u_{i,j+1}^{m-1} u_{i+1,j+1}^{2} + 2\beta m u_{i,j+1}^{m-1} u_{i+1,j+1} u_{i-1,j+1} - \beta m u_{i,j+1}^{m-1} u_{i-1,j+1}^{2} - u_{i,j} \tag{5}$$

where

$$\underset{\sim}{u}_{j+1} = \left(u_{1,j+1}, u_{2,j+1}, ..., u_{n,j+1}\right).$$

Eq. (5) forms a system of nonlinear equations at each time level $j+1$ by considering all interior grid points in Fig. 1 which can be written as

$$F\left(\underset{\sim}{u}_{j+1}\right) = 0 \tag{6}$$

By using the Newton method, Eq. (6) can be linearized in order to construct the corresponding linear system as

$$J\left(\underset{\sim}{u}_{j+1}^{(k)}\right) \Delta \underset{\sim}{h}_{j+1}^{(k)} = -F\left(\underset{\sim}{u}_{j+1}^{(k)}\right) \tag{7}$$

where

$$J\left(\underset{\sim}{u}_{j+1}^{(k)}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial u_{1,j+1}} & \frac{\partial f_1}{\partial u_{2,j+1}} & \cdots & \frac{\partial f_1}{\partial u_{n,j+1}} \\ \frac{\partial f_2}{\partial u_{1,j+1}} & \frac{\partial f_2}{\partial u_{2,j+1}} & \cdots & \frac{\partial f_2}{\partial u_{n,j+1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_{1,j+1}} & \frac{\partial f_n}{\partial u_{2,j+1}} & \cdots & \frac{\partial f_n}{\partial u_{n,j+1}} \end{bmatrix}, \quad \Delta \underset{\sim}{h}_{j+1}^{(k)} = \begin{bmatrix} \Delta h_{1,j+1} \\ \Delta h_{2,j+1} \\ \vdots \\ \Delta h_{n,j+1} \end{bmatrix}.$$

Lastly, the numerical solution for the 1D PME problems can be computed iteratively by using the following expression.

$$\underset{\sim}{u}_{j+1}^{(k+1)} = \underset{\sim}{u}_{j+1}^{(k)} + \Delta \underset{\sim}{h}_{j+1}^{(k)} \tag{8}$$

where k is the iteration number.

2.2 Formulation and Implementation of Four-Point EGSOR method

Let consider the linear system in Eq. (7) be rewritten in general form as

$$A\underset{\sim}{\Delta h} = \underset{\sim}{b} \tag{9}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{(n\times n)}, \quad \underset{\sim}{\Delta h} = \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \vdots \\ \Delta h_n \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

There are various methods have been proposed to solve as well as to accelerate the convergence of iteration procedure for linear system shown in Eq. (9) (Young, 1954; Saad, 2003; Young, 2014). By decomposing matrix A into D, L and V which are the diagonal, lower triangular part and upper triangular part of the matrix respectively as

$$A = D - L - V, \tag{10}$$

the formulation of SOR method that uses to solve Eq. (9) can be obtained as

$$\underset{\sim}{\Delta h}^{(k+1)} = (D - \omega L)^{-1}(\omega V - (1-\omega)D)\underset{\sim}{\Delta h}^{(k)} + \omega(D - \omega L)^{-1}\underset{\sim}{b}. \tag{11}$$

In addition to that, Evans (1985) has proposed a four-point block iterative method which is also known as Explicit Group (EG) in solving large linear systems. In accordance to the aim of this paper, the Four-Point EGSOR iterative method is applied for solving the generated linear system. Thus, to derive the proposed iterative method, together with the reference of Fig. 1, let considers a group of four points be used to form a linear system (4×4) as follows.

$$\begin{bmatrix} a_{i,i} & a_{i,i+1} & a_{i,i+2} & a_{i,i+3} \\ a_{i+1,i} & a_{i+1,i+1} & a_{i+1,i+2} & a_{i+1,i+3} \\ a_{i+2,i} & a_{i+2,i+1} & a_{i+2,i+2} & a_{i+2,i+3} \\ a_{i+3,i} & a_{i+3,i+1} & a_{i+3,i+2} & a_{i+3,i+3} \end{bmatrix} \begin{bmatrix} \Delta h_i \\ \Delta h_{i+1} \\ \Delta h_{i+2} \\ \Delta h_{i+3} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}, \tag{12}$$

where

$$S_t = b_{i+t-1} - \sum_{j=1}^{i-1} a_{i+t-1,j}\Delta h_j^{(k+1)} - \sum_{j=i+2}^{n} a_{i+t-1,j}\Delta h_j^{(k)}, \quad t = 1, 2, 3, 4$$

Eq. (12) can be easily inverted and derived into a Four-Point EGSOR iterative method as

$$\begin{bmatrix} \Delta h_i \\ \Delta h_{i+1} \\ \Delta h_{i+2} \\ \Delta h_{i+3} \end{bmatrix}^{(k+1)} = (1-\omega)\begin{bmatrix} \Delta h_i \\ \Delta h_{i+1} \\ \Delta h_{i+2} \\ \Delta h_{i+3} \end{bmatrix}^{(k)} + \omega \begin{bmatrix} a_{i,i} & a_{i,i+1} & a_{i,i+2} & a_{i,i+3} \\ a_{i+1,i} & a_{i+1,i+1} & a_{i+1,i+2} & a_{i+1,i+3} \\ a_{i+2,i} & a_{i+2,i+1} & a_{i+2,i+2} & a_{i+2,i+3} \\ a_{i+3,i} & a_{i+3,i+1} & a_{i+3,i+2} & a_{i+3,i+3} \end{bmatrix}^{-1} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} \tag{13}$$

Noticed that when $\omega = 1$, Eq. (13) can be reduced into a Four-Point EG iterative method. Then, by using Eq. (13), the Four-Point EGSOR algorithm can be constructed into Algorithm 1 (Saudi and Sulaiman, 2012).

Algorithm 1. Four-point EGSOR

i. Initialize $\underset{\sim}{u}^{(0)} \leftarrow 1.0000, \varepsilon_{Newton} \leftarrow 10^{-10}, \varepsilon \leftarrow 10^{-10}$.
ii. For $j = 0, 1, 2, \ldots, s$, implement
   a. Set $\Delta \underset{\sim}{h}^{(0)} = 0$,
   b. Calculate the matrix A and the vector $\underset{\sim}{b}$,
   c. For $i = 1, 5, \ldots, n-6$, compute Eq. (13) iteratively,
   d. For $i = n-2$, compute the ungroup case (Evans and Abdullah, 1985).
iii. Convergence test $\left| \Delta \underset{\sim}{h}^{(k+1)} - \Delta \underset{\sim}{h}^{(k+1)} \right| \le \varepsilon$. If converges, go to (iv). Otherwise back to (ii).
iv. Compute Eq. (8) to obtain the approximate solutions.
v. Convergence test $\left| F\left(\underset{\sim}{u}^{(k+1)}\right) - F\left(\underset{\sim}{u}^{(k)}\right) \right| \le \varepsilon_{Newton}$. If converges, $j+1$. Otherwise back to (i).
vi. Display approximate solutions.

The estimate values of the relaxation parameter ω are determined within range ±0.01 by running Algorithm 1 several times and the value that gives the least number of iterations is said to be the optimal ω.

2.3. Numerical Experiments

In order to verify the efficiency of the Four-Point Newton-EGSOR iterative method, four selected 1D PME problems are used. The numerical results will be compared with the Four-Point Newton-EG and the Newton-GS iterative methods. For the comparison purpose, three criteria will be considered namely number of iterations (k), computational time which is measured in seconds (Time) and maximum absolute error (Error). In addition to that, tolerance error for convergence is $\varepsilon = 10^{-10}$. The followings are four examples of 1D PME problems.

Example 1 (Polyanin and Zaitsev, 2004)
Consider $K = 1$ and $m = 1$ in Eq. (1) gives

$$\frac{du}{dt} = \frac{d}{dx}\left(u\frac{du}{dx}\right), \qquad u(x,0) = x. \tag{14}$$

The exact solution is

$$u(x,t) = x + t. \tag{15}$$

Example 2 (Polyanin and Zaitsev, 2004)
Consider $K = 0.5$ and $m = -1$ in Eq. (1) gives

$$\frac{du}{dt} = 0.5\frac{d}{dx}\left(u^{-1}\frac{du}{dx}\right), \quad u(x,0) = \frac{1}{0.6x + 1.3}. \tag{16}$$

The given exact solution is

$$u(x,t)=\frac{1}{0.6x-0.18t+1.3}. \tag{17}$$

Example 3 (Wazwaz, 2007)
Consider $K=1$ and $m=2$ in Eq. (1) gives

$$\frac{du}{dt}=\frac{d}{dx}\left(u^{2}\frac{du}{dx}\right), \qquad u(x,0)=\frac{x+1}{4}. \tag{18}$$

and the exact solution is

$$u(x,t)=\frac{x+1}{2\sqrt{4-t}}. \tag{19}$$

Example 4 (Wazwaz, 2007)
Consider $K=0.5$ and $m=-2$ in Eq. (1) gives

$$\frac{du}{dt}=0.5\frac{d}{dx}\left(u^{-2}\frac{du}{dx}\right), \quad u(x,0)=\frac{1}{\sqrt{0.7x+1.35}}. \tag{20}$$

and satisfy the exact solution given by

$$u(x,t)=\frac{1}{\sqrt{0.7x-0.1225t+1.35}}. \tag{21}$$

The numerical results obtained have been summarized in Table 1, 2 and 3.

Table 1. Comparison of number of iterations (k), computational time in seconds (Time) and maximum absolute errors (Error) for the iterative methods using Examples 1 and 2.

d	Method	Example 1			Example 2		
		k	Time	Error	k	Time	Error
64	GS	3835	2.38	2.76E-08	1720	1.13	2.03E-05
	EG	1109	0.34	5.88E-09	504	0.55	2.03E-05
	EGSOR	263	0.12	3.43E-11	175	0.16	2.03E-05
128	GS	13678	7.50	1.22E-07	6034	4.06	2.02E-05
	EG	3899	1.65	2.64E-08	1718	1.43	2.03E-05
	EGSOR	513	0.34	4.28E-11	337	0.41	2.03E-05
256	GS	48395	38.58	5.33E-07	20907	27.03	2.00E-05
	EG	13799	11.20	1.10E-07	5976	11.25	2.02E-05
	EGSOR	1010	1.37	6.90E-11	656	1.35	2.03E-05
512	GS	169693	252.94	2.10E-06	71385	287.34	1.93E-05
	EG	48666	77.31	4.99E-07	20701	84.57	2.00E-05
	EGSOR	2027	5.33	7.69E-10	1297	4.46	2.03E-05
1024	GS	587031	1712.49	7.62E-06	239975	1741.01	1.72E-05

	EG	170300	557.86	2.08E-06	70888	571.03	1.94E-05
	EGSOR	4072	22.43	1.54E-10	2477	16.63	2.03E-05

Table 2. Comparison of number of iterations (k), computational time in seconds (Time) and maximum absolute errors (Error) for the iterative methods using Examples 3 and 4.

d	Method	Example 3			Example 4		
		k	Time	Error	k	Time	Error
64	GS	1344	1.17	8.39E-05	2015	1.26	2.88E-06
	EG	402	0.38	8.39E-05	592	0.33	2.89E-06
	EGSOR	216	0.13	8.39E-05	191	0.16	2.90E-06
128	GS	4824	2.84	8.39E-05	7082	4.90	2.90E-06
	EG	1361	1.00	8.39E-05	2033	1.85	2.94E-06
	EGSOR	437	0.41	8.39E-05	380	0.44	2.96E-06
256	GS	17308	20.03	8.39E-05	24325	45.42	2.71E-06
	EG	4836	6.77	8.39E-05	7007	12.11	2.92E-06
	EGSOR	873	1.47	8.39E-05	734	1.42	2.97E-06
512	GS	61658	270.11	8.40E-05	81729	354.79	1.86E-06
	EG	17333	46.85	8.39E-05	23769	93.86	2.73E-06
	EGSOR	1718	5.38	8.38E-05	1428	4.86	2.98E-06
1024	GS	218147	2008.35	8.43E-05	265698	2293.23	3.33E-06
	EG	61779	342.02	8.40E-05	79057	733.85	1.89E-06
	EGSOR	3344	20.49	8.39E-05	2881	18.75	2.97E-06

Table 3. Reduction in percentages of the iterative methods compared with GS method.

d	Method	Number of iterations				Computational time			
		Example 1	Example 2	Example 3	Example 4	Example 1	Example 2	Example 3	Example 4
64	EG	71.08%	70.70%	70.09%	70.62%	85.71%	51.33%	67.52%	73.81%
	EGSOR	93.14%	89.83%	83.93%	90.52%	94.96%	85.84%	88.89%	87.30%
128	EG	71.49%	71.53%	71.79%	71.29%	78.00%	64.78%	64.79%	62.24%
	EGSOR	96.25%	94.41%	90.94%	94.63%	95.47%	89.90%	85.56%	91.02%
256	EG	71.49%	71.42%	72.06%	71.19%	70.97%	58.38%	66.20%	73.34%
	EGSOR	97.91%	96.86%	94.96%	96.98%	96.45%	95.01%	92.66%	96.87%
512	EG	71.32%	71.00%	71.89%	70.92%	69.44%	70.57%	82.66%	73.54%
	EGSOR	98.81%	98.18%	97.21%	98.25%	97.89%	98.45%	98.01%	98.63%
1024	EG	70.99%	70.46%	71.68%	70.25%	67.42%	67.20%	82.97%	68.00%
	EGSOR	99.31%	98.97%	98.47%	98.92%	98.69%	99.04%	98.98%	99.18%

3. Conclusion

In this paper, the efficiency of the proposed Four-Point Newton-EGSOR iterative method for the implicit finite difference solution of 1D PME has been demonstrated by using four PME problems with $m = 1, -1, 2$ and -2. The numerical results obtained are compared to the other two iterative methods, i.e. Four-Point Newton-EG and Newton-GS iterative methods. As presented in Tables 1 and 2, the proposed iterative method showed its superiority in term of the number of iterations and computational time compared to the other two iterative methods. With the Newton-GS as a control, the Four-Point Newton-EGSOR has reduced number of iterations approximately 83.93% - 99.31% and computational time approximately 85.56% - 99.18%, refer to Table 3. All three iterative methods showed good agreement in term of accuracy. With these results, it can be said that the Four-Point Newton-EGSOR iterative method can be a promising numerical method in tackling nonlinear differential equation problems. To enhance the rate of convergence of the current method, in future work, this study will investigate the application of MSOR as in Sulaiman *et al.* (2012).

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